Problem 2.2

Find the electric field (magnitude and direction) a distance z above the midpoint between equal and opposite charges ($\pm q$), a distance d apart (same as Ex. 2.1, except that the charge at $x = +d/2$ is $-q$).

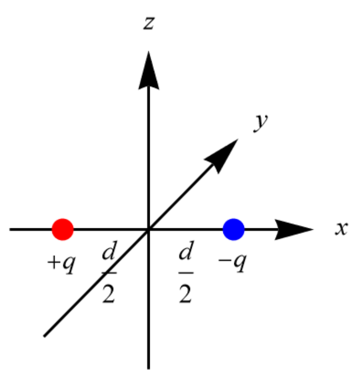
Solution

Derivation

Start with Gauss's law, one of the governing equations for the electric field.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Normally to determine a vector field one needs to know the curl in addition to the divergence, but because of symmetry (to be explained later) the divergence is sufficient. The aim is to solve this equation for $\mathbf{E}$ for the two charges shown below.



The point charges, $+q$ and $-q$, are located at $(-d/2, 0, 0)$ and $(d/2, 0, 0)$, respectively, so their charge densities can be expressed using Dirac delta functions.

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{1}{\epsilon_0}\left[q\delta\left(x + \frac{d}{2}\right)\delta(y)\delta(z) - q\delta\left(x - \frac{d}{2}\right)\delta(y)\delta(z)\right], \quad -\infty < x, y, z < \infty \\ &= \frac{q}{\epsilon_0}\delta\left(x + \frac{d}{2}\right)\delta(y)\delta(z) - \frac{q}{\epsilon_0}\delta\left(x - \frac{d}{2}\right)\delta(y)\delta(z)\end{aligned}$$

Make the substitution $\mathbf{E} = \mathbf{F} + \mathbf{G}$.

$$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \frac{q}{\epsilon_0}\delta\left(x + \frac{d}{2}\right)\delta(y)\delta(z) - \frac{q}{\epsilon_0}\delta\left(x - \frac{d}{2}\right)\delta(y)\delta(z)$$

$$\nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G} = \frac{q}{\epsilon_0}\delta\left(x + \frac{d}{2}\right)\delta(y)\delta(z) - \frac{q}{\epsilon_0}\delta\left(x - \frac{d}{2}\right)\delta(y)\delta(z)$$

If we set

$$\nabla \cdot \mathbf{F} = \frac{q}{\epsilon_0}\delta\left(x + \frac{d}{2}\right)\delta(y)\delta(z), \quad -\infty < x, y, z < \infty, \tag{1}$$

then the previous equation becomes

$$\nabla \cdot \mathbf{G} = -\frac{q}{\epsilon_0}\delta\left(x - \frac{d}{2}\right)\delta(y)\delta(z), \quad -\infty < x, y, z < \infty. \tag{2}$$

Substituting $\mathbf{E} = \mathbf{F} + \mathbf{G}$ is essentially invoking the principle of superposition: The electric field at $\mathbf{r}$ in the two-charge system is the vector sum of the fields from each charge individually. The reason it works is because the divergence is a linear operator. Away from $\left(-\frac{d}{2}, 0, 0\right)$ the right side of equation (1) is zero.

$$\nabla \cdot \mathbf{F} = 0, \quad (x, y, z) \neq \left(-\frac{d}{2}, 0, 0\right)$$

The electric field around the charge at $\left(-\frac{d}{2}, 0, 0\right)$ is spherically symmetric with respect to this point: $\mathbf{F} = F(\mathscr{r})\hat{\boldsymbol{\mathscr{r}}}$, where $\mathscr{r} = \sqrt{(x + d/2)^2 + (y - 0)^2 + (z - 0)^2}$ is the radial distance from $\left(-\frac{d}{2}, 0, 0\right)$. Expand the divergence operator in spherical coordinates.

$$\frac{1}{\mathscr{r}^2}\frac{d}{d\mathscr{r}}(\mathscr{r}^2 F) = 0$$

Multiply both sides by $\mathscr{r}^2$.

$$\frac{d}{d\mathscr{r}}(\mathscr{r}^2 F) = 0$$

Integrate both sides with respect to $\mathscr{r}$.

$$\mathscr{r}^2 F = C_1$$

Divide both sides by $\mathscr{r}^2$.

$$F(\mathscr{r}) = \frac{C_1}{\mathscr{r}^2}$$

To determine C_1, integrate both sides of equation (1) over the volume of a sphere centered at $\left(-\frac{d}{2}, 0, 0\right)$ with radius ε.

$$\iiint\limits_{\left(x+\frac{d}{2}\right)^2+y^2+z^2\leq\varepsilon^2} \nabla \cdot \mathbf{F}\, d\tau = \iiint\limits_{\left(x+\frac{d}{2}\right)^2+y^2+z^2\leq\varepsilon^2} \frac{q}{\epsilon_0}\delta\left(x + \frac{d}{2}\right)\delta(y)\delta(z)\, d\tau$$

$$= \frac{q}{\epsilon_0}\underbrace{\iiint\limits_{\left(x+\frac{d}{2}\right)^2+y^2+z^2\leq\varepsilon^2} \delta\left(x + \frac{d}{2}\right)\delta(y)\delta(z)\, d\tau}_{=1}$$

Apply the divergence theorem on the left and switch to spherical coordinates $(\mathscr{r}, \phi, \theta)$, where θ is the angle from the polar axis.

$$\oiint\limits_{\left(x+\frac{d}{2}\right)^2+y^2+z^2=\varepsilon^2} \mathbf{F} \cdot d\mathbf{S} = \frac{q}{\epsilon_0}$$

$$\oiint\limits_{\mathscr{r}^2=\varepsilon^2} \mathbf{F} \cdot d\mathbf{S} = \frac{q}{\epsilon_0}$$

$$\oiint\limits_{\mathscr{r}=\varepsilon} [F(\mathscr{r})\hat{\boldsymbol{\mathscr{r}}}] \cdot (\hat{\boldsymbol{\mathscr{r}}}\, dS) = \frac{q}{\epsilon_0}$$

$$\int_0^{\pi}\int_0^{2\pi} F(\varepsilon)(\varepsilon^2 \sin\theta\, d\phi\, d\theta) = \frac{q}{\epsilon_0}$$

Evaluate the double integral and solve for C_1.

$$\begin{aligned}
\varepsilon^2 F(\varepsilon)\left(\int_0^{2\pi} d\phi\right)\left(\int_0^{\pi}\sin\theta\, d\theta\right) &= \frac{q}{\epsilon_0} \\
\varepsilon^2\left(\frac{C_1}{\varepsilon^2}\right)(2\pi)(2) &= \frac{q}{\epsilon_0} \\
C_1(4\pi) &= \frac{q}{\epsilon_0} \\
C_1 &= \frac{q}{4\pi\epsilon_0}
\end{aligned}$$

As a result,

$$F(\mathscr{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2} \qquad \Rightarrow \quad \mathbf{F} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}},$$

or in terms of the original variables,

$$F(x,y,z) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(x+\frac{d}{2}\right)^2+y^2+z^2} \quad \Rightarrow \quad \mathbf{F} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(x+\frac{d}{2}\right)^2+y^2+z^2}\frac{\left\langle x+\frac{d}{2},y,z\right\rangle}{\sqrt{\left(x+\frac{d}{2}\right)^2+y^2+z^2}}.$$

Using the same argument for the charge $-q$ at $\left(\frac{d}{2},0,0\right)$,

$$G(x,y,z) = \frac{-q}{4\pi\epsilon_0}\frac{1}{\left(x-\frac{d}{2}\right)^2+y^2+z^2} \quad \Rightarrow \quad \mathbf{G} = \frac{-q}{4\pi\epsilon_0}\frac{1}{\left(x-\frac{d}{2}\right)^2+y^2+z^2}\frac{\left\langle x-\frac{d}{2},y,z\right\rangle}{\sqrt{\left(x-\frac{d}{2}\right)^2+y^2+z^2}}.$$

Therefore, the electric field of the two-charge system is

$$\begin{aligned}
\mathbf{E} &= \mathbf{F}+\mathbf{G} \\
&= \frac{q}{4\pi\epsilon_0}\frac{\left\langle x+\frac{d}{2},y,z\right\rangle}{\left[\left(x+\frac{d}{2}\right)^2+y^2+z^2\right]^{3/2}} + \frac{-q}{4\pi\epsilon_0}\frac{\left\langle x-\frac{d}{2},y,z\right\rangle}{\left[\left(x-\frac{d}{2}\right)^2+y^2+z^2\right]^{3/2}} \\
&= \frac{q}{4\pi\epsilon_0}\left\{\frac{\left\langle x+\frac{d}{2},y,z\right\rangle}{\left[\left(x+\frac{d}{2}\right)^2+y^2+z^2\right]^{3/2}} - \frac{\left\langle x-\frac{d}{2},y,z\right\rangle}{\left[\left(x-\frac{d}{2}\right)^2+y^2+z^2\right]^{3/2}}\right\}.
\end{aligned}$$

To obtain the electric field a distance z above the midpoint between the charges, set $x=0$ and $y=0$.

$$\mathbf{E}(0,0,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{\left\langle \frac{d}{2},0,z\right\rangle}{\left(\frac{d^2}{4}+z^2\right)^{3/2}} - \frac{\left\langle -\frac{d}{2},0,z\right\rangle}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\right] = \frac{q}{4\pi\epsilon_0}\frac{\langle d,0,0\rangle}{\left(\frac{d^2}{4}+z^2\right)^{3/2}} = \frac{qd}{4\pi\epsilon_0}\frac{\hat{\mathbf{x}}}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}$$

Note that generally the electric field at $\mathbf{r} = \langle x, y, z \rangle$ due to a point charge at $\mathbf{r}' = \langle x', y', z' \rangle$ is

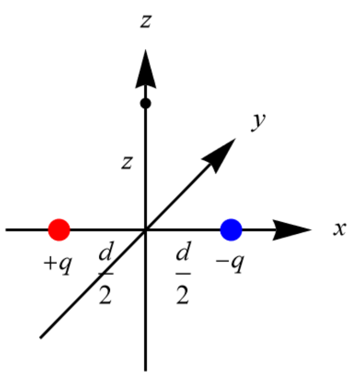
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}},$$

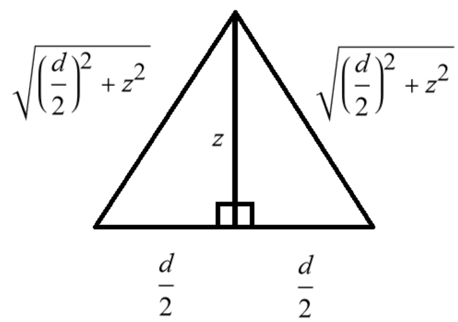
where $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$. This formula applies for a continuous charge distribution in one, two, or three dimensions as follows.

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} \quad \Rightarrow \quad \mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, dl' & \text{for a line charge density} \\ \dfrac{1}{4\pi\epsilon_0}\displaystyle\iint \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, da' & \text{for a surface charge density} \\ \dfrac{1}{4\pi\epsilon_0}\displaystyle\iiint \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, d\tau' & \text{for a volume charge density} \end{cases}$$

Just Using the Formula

The answer will be found again just by using the formula at the top. Start by drawing a schematic of the two charges.





Use the principle of superposition to find the electric field a distance z above the origin. Note that $\mathbf{r}$ is the position vector to where we want to know the electric field, $\mathbf{r}'_i$ is the position vector to charge q_i, and $\boldsymbol{\mathscr{r}}_i = \mathbf{r} - \mathbf{r}'_i$ is the position vector from charge q_i to where we want to know the electric field.

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{2}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i \\ &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{2}\frac{q_i}{\mathscr{r}_i^2}\left(\frac{\mathbf{r} - \mathbf{r}'_i}{\mathscr{r}_i}\right) \\ &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{2}\frac{q_i}{\mathscr{r}_i^3}(\mathbf{r} - \mathbf{r}'_i) \\ &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{2}\frac{q_i}{\mathscr{r}_i^3}\left(\langle 0, 0, z \rangle - \mathbf{r}'_i\right) \end{aligned}$$

Expand the sum, plug in the vectors, and simplify the result.

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{\mathscr{r}_1^3}\left(\langle 0,0,z\rangle - \mathbf{r}_1'\right) + \frac{q_2}{\mathscr{r}_2^3}\left(\langle 0,0,z\rangle - \mathbf{r}_2'\right)\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{+q}{\left(\sqrt{\frac{d^2}{4}+z^2}\right)^3}\left(\langle 0,0,z\rangle - \left\langle -\frac{d}{2},0,0\right\rangle\right) + \frac{-q}{\left(\sqrt{\frac{d^2}{4}+z^2}\right)^3}\left(\langle 0,0,z\rangle - \left\langle \frac{d}{2},0,0\right\rangle\right)\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\left(\langle 0,0,z\rangle - \left\langle -\frac{d}{2},0,0\right\rangle - \langle 0,0,z\rangle + \left\langle \frac{d}{2},0,0\right\rangle\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\langle d,0,0\rangle \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\langle 1,0,0\rangle \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\hat{\mathbf{x}}
\end{aligned}$$

This is the final answer, but it is customary in physics to check limits.

Checking Limits

In the limit as $d \to 0$ and $z \to 0$, for example, the electric field becomes

$$\lim_{d\to 0}\mathbf{E} = \lim_{d\to 0}\frac{1}{4\pi\epsilon_0}\frac{qd}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\hat{\mathbf{x}} = \frac{1}{4\pi\epsilon_0}\frac{q(0)}{\left(\frac{(0)^2}{4}+z^2\right)^{3/2}}\hat{\mathbf{x}} = \mathbf{0}$$

$$\lim_{z\to 0}\mathbf{E} = \lim_{z\to 0}\frac{1}{4\pi\epsilon_0}\frac{qd}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\hat{\mathbf{x}} = \frac{1}{4\pi\epsilon_0}\frac{qd}{\left(\frac{d^2}{4}+(0)^2\right)^{3/2}}\hat{\mathbf{x}} = \frac{1}{4\pi\epsilon_0}\frac{qd}{\frac{d^3}{8}} = \frac{1}{4\pi\epsilon_0}\frac{2q}{\left(\frac{d}{2}\right)^2}\hat{\mathbf{x}}.$$

To see what happens to the electric field as z becomes large, use the binomial theorem to get rid of the fractional exponent. Doing so shows exactly how the electric field goes to zero as z increases.

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{qd}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\hat{\mathbf{x}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{\left[z^2\left(\frac{d^2}{4z^2}+1\right)\right]^{3/2}}\hat{\mathbf{x}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3\left(1+\frac{d^2}{4z^2}\right)^{3/2}}\hat{\mathbf{x}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\left(1+\frac{d^2}{4z^2}\right)^{-3/2}\hat{\mathbf{x}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\left[\sum_{k=0}^{\infty}\frac{\Gamma\left(-\frac{3}{2}+1\right)}{\Gamma(k+1)\Gamma\left(-\frac{3}{2}-k+1\right)}\left(\frac{d^2}{4z^2}\right)^k\right]\hat{\mathbf{x}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\left[\sum_{k=0}^{\infty}\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(k+1)\Gamma\left(-\frac{1}{2}-k\right)}\left(\frac{d}{2z}\right)^{2k}\right]\hat{\mathbf{x}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\left[\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(1)\Gamma\left(-\frac{1}{2}\right)}\left(\frac{d}{2z}\right)^0+\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(2)\Gamma\left(-\frac{3}{2}\right)}\left(\frac{d}{2z}\right)^2+\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(3)\Gamma\left(-\frac{5}{2}\right)}\left(\frac{d}{2z}\right)^4+\cdots\right]\hat{\mathbf{x}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\left[1-\frac{3}{2}\left(\frac{d^2}{4z^2}\right)+\frac{15}{8}\left(\frac{d^4}{16z^4}\right)-\cdots\right]\hat{\mathbf{x}}
\end{aligned}$$

Far away ($z \gg 1$) the higher-order terms in brackets are negligible compared to 1, so the electric field around a dipole falls off as $1/z^3$.